Representations of Right Hereditary Tensor Algebras of Bimodules

Nadiya Gubareni

Institute of Mathematics
Technical University of Częstochowa
Poland

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Representations of algebra

A **representation** of a K-algebra A is an algebra homomorphism $T : A \to \mathrm{End}_K(V)$, where V is a K-vector space.

A K-algebra A is said to be of **finite representation type** (or **finite type**) if A has only a finite number of non-isomorphic finite dimensional indecomposable representations up to isomorphism. Otherwise A is said to be of **infinite representation type**.

Species and tensor algebras

A - a basic ring with Jacobson radical R.
Then $A/R = \prod_{i=1}^{n} F_i$ and $R/R^2 = \prod_{1 \le i,j \le n} {}_iM_j$ with uniquely determined division rings F_i and F_i-F_j-bimodules ${}_iM_j$.

A finite family $\mathfrak{S} = (F_i, {}_iM_j)_{i,j \in I}$ is called a **species**.

$B = \prod\limits_{i \in I} F_i$ is a ring
$M = \bigoplus\limits_{i,j \in I} {}_iM_j$ is a (B, B)-bimodule.
Tensor algebra of the species $\mathfrak{S}$:

$$\mathfrak{T}(\mathfrak{S}) = \mathfrak{T}_B(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n}$$

where $M^{\otimes 0} = B$, $M^{\otimes n} = M^{\otimes (n-1)} \otimes_B M$ for $n > 0$.

Quiver of species

Definition

The **quiver** $\Gamma(\mathfrak{S})$ of a species $\mathfrak{S}$ is defined as the directed graph whose vertices are indexed by the numbers $i = 1, \ldots, n$, and there is an arrow from the vertex i to the vertex j if and only if ${}_iM_j \neq 0$.

Definition

A species $\mathfrak{S}$ is **simply connected** if the underlying graph of $\Gamma(\mathfrak{S})$ is a tree.

Hereditary algebras of f.r.t.

Definition

A ring A is said to be **right hereditary** if all submodules of projective right A-modules are projective.

Theorem (P.Gabriel, V.Dlab, C.M.Ringel)

A finite dimensional K-algebra A is a hereditary algebra of finite type if and only if A is Morita equivalent to a tensor algebra $\mathfrak{T}(\mathfrak{L})$, where $\mathfrak{L}$ is a K-species of finite type.
A is of finite type if and only if diagram of K-species is a finite disjoint union of Dynkin diagrams of the form A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2.

Species of bimodules

Definition (Yu.A.Drozd, 1986)

A **species** is a finite collection $\mathfrak{S} = (A_i, {}_iM_j)_{i,j\in I}$, where
1) all A_i are prime rings
2) all ${}_iM_j$ are A_i-A_j-bimodules.

- If all $A_i = F_i$ are division rings, $\mathfrak{L}$ is a species in the sense of P.Gabriel.
- If all $A_i = D$ are a division ring, $\mathfrak{L}$ is called a D-species

Tensor algebras of species

$$B = \prod_{i \in I} A_i, \quad M = \bigoplus_{i,j \in I} {}_iM_j$$

Tensor algebra of the species $\mathfrak{S}$ is the tensor algebra (B, B)-bimodule M:

$$\mathfrak{T}(\mathfrak{S}) = \bigoplus_{i=0}^{\infty} T_i$$

where $T_0 = B$ and $T_{i+1} = T_i \otimes_B M$ $(i > 0)$.

Discrete valuation rings

A **discrete valuation ring** is a (noncommutative) subring A of a division ring D if there is a (discrete) valuation $v : D \to \mathbf{Z} \bigcup \{\infty\}$ of D such that $A = \{x \in D \ : \ v(x) \geq 0\}$.

Example.
A skew power series ring $K[[x, \sigma]]$ with $xa = \sigma(a)x$ for any $a \in K$, where K is a field and $\sigma : K \to K$ is a nontrivial automorphism of K.

If A is a discrete valuation ring, then:
A is a local ring with non-zero maximal ideal M of the form $M = tA = At$, where $t \in A$ is a non-nilpotent element, and $\bigcap\limits_{i=0}^{\infty} M^i = 0$.

Proposition.

Let $\mathcal{O}_1$, $\mathcal{O}_2$ be distinct discrete valuation rings of a division ring D with maximal ideals M_1 and M_2. Then

- $D = \mathcal{O}_1 + \mathcal{O}_2$.
- There is an element $\pi_1 \in M_1 \setminus M_1^2$ and $\pi_1 \in \mathcal{O}_2^*$. Analogously there is an element $\pi_2 \in M_2 \setminus M_2^2$ and $\pi_2 \in \mathcal{O}_1^*$. So that $M_1 = \pi_1\mathcal{O}_1$, and $M_2 = \pi_2\mathcal{O}_2$.
- $D^* = \mathcal{O}_1^* \cdot \mathcal{O}_2^* = \mathcal{O}_2^* \cdot \mathcal{O}_1^*$.

Proposition.

Let $\mathcal{O}_1$, $\mathcal{O}_2$ be distinct discrete valuation rings of a division ring D, and $A = \mathcal{O}_1 \cap \mathcal{O}_2$. Then there exists an element $x \in D$ such that $x^n \notin A$ and $x^{-n} \notin A$ for any $n \in N$.

$\mathcal{O}$-species

Let

- $\{\mathcal{O}_i\}$ be a family of discrete valuation rings (not necessary commutative) with the radical $\mathcal{M}_i$ and a division ring of fractions D_i for $i = 1, 2, \ldots, k$;
- $\{D_j\}$ for $j = k+1, \ldots, n$ be a family of skew fields.
- $\{n_1, n_2, \ldots, n_k\}$ be a set of natural numbers.
- $$H_{n_i}(\mathcal{O}_i) = \begin{pmatrix} \mathcal{O}_i & \mathcal{O}_i & \cdots & \mathcal{O}_i \\ R_i & \mathcal{O}_i & \cdots & \mathcal{O}_i \\ \vdots & \vdots & \ddots & \vdots \\ R_i & R_i & \cdots & \mathcal{O}_i \end{pmatrix}$$

$\mathcal{O}$-species

Definition

Let $I = \{1, 2, \ldots, n\}$. An $\mathcal{O}$-**species** is a family $\Omega = (A_i, {}_iM_j)_{i,j \in I}$, where

- $A_i = H_{n_i}(\mathcal{O}_i)$, for $i = 1, \ldots, k$,
- $A_j = D_j$ for $j = k+1, \ldots, n$,
- each ${}_iM_j$ is an $(\tilde{A}_i, \tilde{A}_j)$-bimodule that is finite dimensional both as a left D_i-vector space and as a right D_j-vector space.

An $\mathcal{O}$-species Ω is said to be a $(D, \mathcal{O})$-**species** if all $D_i = D$.

Representations of $\mathcal{O}$-species

Definition

A **representation** $V = (U_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ of an $\mathcal{O}$-species $\Omega = (A_i, {}_iM_j)_{i,j\in I}$ is a family of right A_i-modules U_i $(i = 1, 2, \dots, k)$, a family of right D_r-vector spaces V_r $(r = k+1, \dots, n)$, and of D_j-linear maps:

$$ {}_j\varphi_i : U_i \otimes_{A_i} {}_iM_j \longrightarrow V_j \tag{2.3}$$

for each $i = 1, 2, \dots, k$ and $j = k+1, \dots, n$, and

$$ {}_j\psi_r : V_r \otimes_{D_r} {}_rM_j \longrightarrow V_j \tag{2.4}$$

for each $r, j = k+1, \dots, n$.

$\mathrm{Rep}(\Omega)$ and $\mathrm{Mod}_r\mathfrak{T}(\Omega)$

Proposition.

Let Ω be a simply connected $\mathcal{O}$-species. Then the category $\mathrm{Rep}(\Omega)$ of all representations of Ω and the category $\mathrm{Mod}_r\mathfrak{T}(\Omega)$ of all right $\mathfrak{T}(\Omega)$-modules are naturally equivalent.

Rings of bounded representation type

Definition

An A-module M is called **finitely presented** if there is an epimorphism $\varphi : A^{(n)} \to M$ such that $\mathrm{Ker}(\varphi)$ is a finitely generated A-module.

Definition (R.B.Warfield,Jr)

A ring A is of **bounded representation type** (b.r.t., for short) if the number of generators required for indecomposable finitely presented right A-modules is bounded.

$\mathcal{O}$-species of bounded representation type

Definition

An $\mathcal{O}$-species Ω is said to be of **bounded representation type** if the dimensions of its indecomposable finite dimensional representations are bounded above.

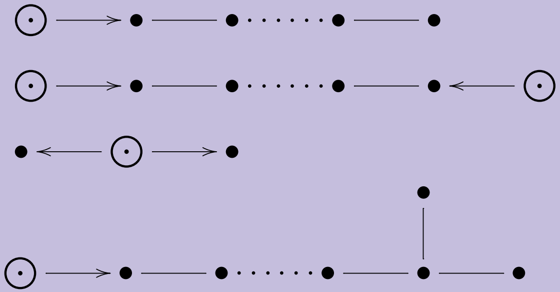
Proposition.

A simply connected $\mathcal{O}$-species Ω is of bounded representation type if and only if the tensor algebra $\mathfrak{T}(\Omega)$ is of bounded representation type.

$(D, \mathcal{O})$-species of b.r.t.

Theorem

Let $\mathcal{O}_i$ be discrete valuation rings with a common skew field of fractions D. Then a simply connected $(D, \mathcal{O})$-species Ω is of b.r.t. iff the diagram $Q(\Omega)$ is a finite disjoint union of Dynkin diagrams of the forms A_n, D_n, E_6, E_7, E_8 and the following diagrams:



Flat mixed matrix problem

Let $\Delta = \{\mathcal{O}_i\}_{i=1,\dots,k}$, D - a common division ring of fractions of $\mathcal{O}_i$, and $F_{i_s} \in \Delta \cup D$, $\mathbf{T} \in M_{s\times r}(D)$:

$$\mathbf{T} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{T}_{11} & \cdots & \mathbf{T}_{1j} & \cdots & \mathbf{T}_{1m} \\ \hline \vdots & \ddots & \vdots & \ddots & \vdots \\ \hline \mathbf{T}_{i1} & \cdots & \mathbf{T}_{ij} & \cdots & \mathbf{T}_{im} \\ \hline \vdots & \ddots & \vdots & \ddots & \vdots \\ \hline \mathbf{T}_{n1} & \cdots & \mathbf{T}_{nj} & \cdots & \mathbf{T}_{nm} \\ \hline \end{array}$$

Flat mixed matrix problems

The admissible transformations with the matrix $\mathbf{T}$:

- Left F_{i_s}-elementary transformations of rows within the strip $\mathbf{T}_i$.
- Right F_{j_t}-elementary transformations of rows within the strip $\mathbf{T}^j$.
- Additions of rows in the strip $\mathbf{T}_j$ multiplied on the left by elements of $F_r \in \Delta \cup D$ to rows in the strip $\mathbf{T}_i$.
- Additions of columns in the strip $\mathbf{T}^i$ multiplied on the right by elements of $F_p \in \Delta \cup D$ to columns in the strip $\mathbf{T}^j$.

Matrix problem of b.r.t.

Definition

A flat matrix problem is of **bounded representation type** if there is a constant C such that $\dim(\mathbf{X}) < C$ for all indecomposable matrices $\mathbf{X}$.
Otherwise it is of **unbounded representation type**.

Lemma.

A simply connected $(D, \mathcal{O})$-species Ω is of bounded representation type if and only if the corresponding matrix problem is of bounded representation type.

Example of a matrix problem of b.r.t.

Matrix problem I.

$\mathcal{O}$ - distinct discrete valuation ring with a division ring of fractions D; $\mathbf{T} \in M_{k\times s}(D)$

Admissible transformations on $\mathbf{T}$:
1. Left $\mathcal{O}$-elementary transformations of rows within the matrix $\mathbf{T}$.
2. Right $\mathcal{O}$-elementary transformations of columns within the matrix $\mathbf{T}$.

$$\mathbf{T} = \left(\begin{array}{cccc|c} \pi^{n_1} & 0 & \cdots & 0 & \\ 0 & \pi^{n_2} & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \mathbf{O} \\ 0 & 0 & \cdots & \pi^{n_k} & \\ \hline & & \mathbf{O} & & \mathbf{O} \end{array}\right)$$

Example of a matrix problem of b.r.t.

Matrix problem II.

$\mathcal{O}_1, \mathcal{O}_2$ - distinct discrete valuation rings with a common division ring of fractions D; $\mathbf{T} \in M_{k\times s}(D)$

Admissible transformations on $\mathbf{T}$:
1. Left $\mathcal{O}_1$-elementary transformations of rows within the matrix $\mathbf{T}$.
2. Right $\mathcal{O}_2$-elementary transformations of columns within the matrix $\mathbf{T}$.

$$\mathbf{T} = \left(\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right),$$

where $\mathbf{I}$ is the identity matrix.

Examples of matrix problems of ub.r.t.

Matrix problem III.

$\mathcal{O}_1, \mathcal{O}_2$ - distinct discrete valuation rings with a common division ring of fractions D. $\mathbf{T} \in M_{k\times s}(D)$

Admissible transformations on $\mathbf{T}$:
1. Left $\mathcal{O}_1 \bigcap \mathcal{O}_2$-elementary transformations of rows within the matrix $\mathbf{T}$.
2. Right D-elementary transformations of columns within the matrix $\mathbf{T}$.

Examples of matrix problems of ub.r.t.

Lemma.

The matrix problem III is of unbounded representation type:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ x & x^2 & \cdots & x^n \end{pmatrix}$$

where $x \notin \mathcal{O}_1 \bigcap \mathcal{O}_2$ and $x^{-1} \notin \mathcal{O}_1 \bigcap \mathcal{O}_2$.

Examples of matrix problems of ub.r.t.

Matrix problem IV.

$\mathcal{O}$ - a DVR with division ring of fractions D, $\varepsilon_1, \dots, \varepsilon_{n-1} \in \mathcal{O}$; $\mathbf{T} \in M_{k\times s}(D)$

$$\mathbf{T} = \begin{array}{|c|c|c|} \hline \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \hline \end{array}$$

Admissible transformations on $\mathbf{T}$:

1. Left elementary $\mathcal{O}$-transformations of rows on the whole matrix $\mathbf{T}$.
2. Right D-elementary transformations of columns within each vertical strip $\mathbf{A}_i$ ($i = 1, 2, 3$).
3. Additions of any column of the block $\mathbf{A}_1$ multiplied on the right by an arbitrary element of D to any column of the block $\mathbf{A}_2$.

Examples of matrix problems of ub.r.t.

Matrix problem V.

$\mathcal{O}_1, \mathcal{O}_2$ - distinct discrete valuation rings with a common division ring of fractions D.

$$\mathbf{T} = \boxed{\mathbf{A}_1 \mid \mathbf{A}_2}$$

Admissible transformations on $\mathbf{T}$:
1. Left $\mathcal{O}_1$-elementary transformations of rows within the matrix $\mathbf{T}$.
2. Right $\mathcal{O}_2$-elementary transformations of rows within the matrix $\mathbf{A}_1$.
2. Right D-elementary transformations of columns within the matrix $\mathbf{A}_2$.

Thank you very much for your attention.